Reordenación de series armónicas alternadas

Se tiene que

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \ln(n) + \gamma + \varepsilon_n$$

donde γ = 0.5772156649... (se desconoce si es irracional, aunque se supone que es trascendente), y $\lim_{n\to\infty} \varepsilon_n = 0$. Sean

$$HI_n = \sum_{k=1}^{n/2} \frac{1}{2k-1} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{n-1} \quad (n\ par)$$

$$HP_n = \sum_{k=1}^{n/2} \frac{1}{2k} = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{n} \quad (n\ par)$$

Se tiene que

$HP_{2n} = \frac{1}{2} H_n = \frac{1}{2} \ln(n) + \frac{1}{2} \gamma + \frac{1}{2} \varepsilon_n$
$HI_{2n} = H_{2n} - HP_{2n} = H_{2n} - \frac{1}{2} H_n = \ln(2n) - \frac{1}{2} \ln(n) + \frac{1}{2} \gamma + (\varepsilon_{2n} - \frac{1}{2} \varepsilon_n)$
$HI_{2n} = \ln(2) + \frac{1}{2} \ln(n) + \frac{1}{2} \gamma + (\varepsilon_{2n} - \frac{1}{2} \varepsilon_n)$

$$H^*_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n} = H_{2n} - 2HP_{2n} = H_{2n} - H_n = \ln(2n) - \ln(n) + (\varepsilon_{2n} - \varepsilon_n) = \ln(2) + (\varepsilon_{2n} - \varepsilon_n) \Rightarrow$$

$$H^* = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \lim_{n\to\infty} H^*_{2n} = \ln(2)$$

Entonces

$$H_n(p,q) = 1 + \frac{1}{3} + \ldots + \frac{1}{2p-1} - \frac{1}{2} - \frac{1}{4} - \ldots - \frac{1}{2q} + \frac{1}{2p+1} + \ldots + \frac{1}{4p-1} - \frac{1}{2q+2} - \ldots - \frac{1}{4q} + \ldots - \frac{1}{2qn}$$

$$= HI_{2pn} - HP_{2qn} = \ln(2) + \frac{1}{2}\ln(pn) - \frac{1}{2}\ln(qn) + \varepsilon_{\ldots} = \ln(2) + \frac{1}{2}\ln\left(\frac{p}{q}\right) + \varepsilon_{\ldots} \Rightarrow$$

$$H(p,q) = \ln(2) + \frac{1}{2}\ln\left(\frac{p}{q}\right) = \frac{1}{2}\ln\left(\frac{4p}{q}\right) \qquad (H^* = H(1,1))$$