Desarrollo en fracción continua de $\sqrt{m^2+k}$, con $2m/k$ entero

$$\sqrt{m^2+k} = m + (\sqrt{m^2+k} - m) = m + \cfrac{1}{\cfrac{1}{\sqrt{m^2+k}-m}} = m + \cfrac{1}{\cfrac{\sqrt{m^2+k}+m}{k}} = m + \cfrac{1}{\cfrac{2m+\sqrt{m^2+k}-m}{k}} =$$

$$m + \cfrac{1}{2m/k + \cfrac{\sqrt{m^2+k}-m}{k}} = m + \cfrac{1}{2m/k + \cfrac{1}{\cfrac{k}{\sqrt{m^2+k}-m}}} = m + \cfrac{1}{2m/k + \cfrac{1}{\cfrac{k(\sqrt{m^2+k}+m)}{k}}} =$$

$$m + \cfrac{1}{2m/k + \cfrac{1}{2m+\sqrt{m^2+k}-m}} = m + \cfrac{1}{2m/k + \cfrac{1}{2m + \cfrac{1}{2m/k + \cfrac{1}{2m + \cfrac{1}{2m/k + \cfrac{1}{2m+\dots}}}}}} = \left[m;\ \overline{2m/k,\ 2m}\right]$$

Casos particulares: m=3

k=1: $\sqrt{m^2+1} = \left[m; \overline{2m, 2m}\right] = \left[m; \overline{2m}\right]$ $\qquad \sqrt{10} = \left[3; \overline{6}\right]$

k=2: $\sqrt{m^2+2} = \left[m; \overline{m, 2m}\right]$ $\qquad \sqrt{11} = \left[3; \overline{3, 6}\right]$

k=m: $\sqrt{m^2+m} = \left[m; \overline{2, 2m}\right]$ $\qquad \sqrt{12} = \left[3; \overline{2, 6}\right]$

k=2m: $\sqrt{m^2+2m} = \sqrt{(m+1)^2-1} = \left[m; \overline{1, 2m}\right]$ $\qquad \sqrt{15} = \left[3; \overline{1, 6}\right]$

Si m es par, m=4

k=4: $\sqrt{m^2+4} = \left[m; \overline{m/2, 2m}\right]$ $\qquad \sqrt{20} = \left[4; \overline{2, 8}\right]$

k=m/2: $\sqrt{m^2+m/2} = \left[m; \overline{4, 2m}\right]$ $\qquad \sqrt{18} = \left[4; \overline{4, 8}\right]$

Para k = 1, como la longitud del período es impar, las ecuaciones $\boldsymbol{x^2 - (m^2+1)y^2 = -1}$ tienen soluciones.

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\sqrt{(m+1)^2-k}$, con $p=\frac{2(m+1)}{k}$ entero > 1

Haciendo $Q=(m+1)^2-k=m^2+2m+1-k$, tenemos

$$\sqrt{Q}-m=\frac{1}{\frac{1}{\sqrt{Q}-m}}=\frac{1}{\frac{\sqrt{Q}+m}{2m+1-k}}=\frac{1}{1+\frac{\sqrt{Q}-(m+1-k)}{2m+1-k}}=$$

$$\frac{1}{1+\frac{1}{\frac{2m+1-k}{\sqrt{Q}-(m+1-k)}}}=\frac{1}{1+\frac{1}{\frac{(2m+1-k)\left(\sqrt{Q}+(m+1-k)\right)}{k(2m+1-k)}}}=\frac{1}{1+\frac{1}{\frac{\sqrt{Q}+(m+1-k)}{k}}}=$$

$$\frac{1}{1+\frac{1}{(p-2)+\frac{\sqrt{Q}-(m+1-k)}{k}}}=\frac{1}{1+\frac{1}{(p-2)+\frac{1}{\frac{k\left(\sqrt{Q}+(m+1-k)\right)}{k(2m+1-k)}}}}=$$

$$\frac{1}{1+\frac{1}{(p-2)+\frac{1}{1+\frac{\sqrt{Q}-m}{(2m+1-k)}}}}=\frac{1}{1+\frac{1}{(p-2)+\frac{1}{1+\frac{1}{\frac{(2m+1-k)\left(\sqrt{Q}+m\right)}{2m+1-k}}}}}=$$

$$\frac{1}{1+\frac{1}{(p-2)+\frac{1}{1+\frac{1}{2m+\left(\sqrt{Q}-m\right)}}}}\quad\Rightarrow\quad\sqrt{(m+1)^2-k}=\left[m;\overline{1,\frac{2(m+1)}{k}-2,1,2m}\right]$$

Casos particulares: m=4

k=1: $\sqrt{(m+1)^2-1}=\left[m;\overline{1,2m,1,2m}\right]=\left[m;\overline{1,2m}\right]$ $\quad\sqrt{24}=\left[4;\overline{1,8}\right]$

k=2: $\sqrt{(m+1)^2-2}=\left[m;\overline{1,m-1,1,2m}\right]$ $\quad\sqrt{23}=\left[4;\overline{1,3,1,8}\right]$

k=m+1: $\sqrt{(m+1)^2-(m+1)}=\left[m;\overline{1,0,1,2m}\right]=\left[m;\overline{2,2m}\right]$ $\quad\sqrt{20}=\left[4;\overline{2,8}\right]$

Si m es impar,

k=4: $\sqrt{(m+1)^2-4}=\left[m;\overline{1,\frac{m-3}{2},1,2m}\right]$; $\quad$ m=5: $\sqrt{32}=\left[5;\overline{1,1,1,10}\right]$

k=$\frac{m+1}{2}$: $\sqrt{(m+1)^2-\frac{m+1}{2}}=\left[m;\overline{1,2,1,2m}\right]$; $\quad$ m=5: $\sqrt{33}=\left[5;\overline{1,2,1,10}\right]$

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\sqrt{n^2+4}$, con $n = 2\cdot m+1$

Haciendo $Q = (2m+1)^2 + 4 = 4m^2 + 4m + 5$, tenemos

$$\sqrt{Q}-(2m+1) = \frac{1}{\dfrac{1}{\sqrt{Q}-(2m+1)}} = \frac{1}{\dfrac{\sqrt{Q}+(2m+1)}{4}} = \frac{1}{m+\dfrac{\sqrt{Q}-(2m-1)}{4}} = \frac{1}{m+\dfrac{1}{\dfrac{4}{\sqrt{Q}-(2m-1)}}} =$$

$$\frac{1}{m+\dfrac{1}{\dfrac{4\left(\sqrt{Q}+(2m-1)\right)}{8m+4}}} = \frac{1}{m+\dfrac{1}{\dfrac{\sqrt{Q}+(2m-1)}{2m+1}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{\sqrt{Q}-2}{2m+1}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{1}{\dfrac{2m+1}{\sqrt{Q}-2}}}} =$$

$$\frac{1}{m+\dfrac{1}{1+\dfrac{1}{\dfrac{(2m+1)\left(\sqrt{Q}+2\right)}{(2m+1)^2}}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{1}{\dfrac{\sqrt{Q}+2}{2m+1}}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{1}{1+\dfrac{\sqrt{Q}-(2m-1)}{2m+1}}}} =$$

$$\frac{1}{m+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{\dfrac{2m+1}{\sqrt{Q}-(2m-1)}}}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{\dfrac{(2m+1)\left(\sqrt{Q}+(2m-1)\right)}{8m+4}}}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{\dfrac{\sqrt{Q}+(2m-1)}{4}}}}} =$$

$$\frac{1}{m+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{m+\dfrac{\sqrt{Q}-(2m+1)}{4}}}}} = \frac{1}{m+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{m+\dfrac{1}{\dfrac{4}{\sqrt{Q}-(2m+1)}}}}}} =$$

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$$\cfrac{1}{m+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{m+\cfrac{1}{\cfrac{4\left(\sqrt{Q}+(2m+1)\right)}{4}}}}}} = \cfrac{1}{m+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{m+\cfrac{1}{\sqrt{Q}+(2m+1)}}}}} =$$

$$\cfrac{1}{m+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{m+\cfrac{1}{2(2m+1)+\left(\sqrt{Q}-(2m+1)\right)}}}}} \Rightarrow$$

$$\sqrt{Q} = (2m+1)+\cfrac{1}{m+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{m+\cfrac{1}{2(2m+1)+\cfrac{1}{m+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{m+\cfrac{1}{2(2m+1)+\dots}}}}}}}}}}$$

$$\sqrt{(2m+1)^2+4} = \left[2m+1;\ \overline{m,\ 1,\ 1,\ m,\ 2(2m+1)}\right]$$

$$m=1: \quad \sqrt{(2\cdot 1+1)^2+4} = \sqrt{13} = \left[3;\ \overline{1,\ 1,\ 1,\ 1,\ 6}\right]$$

$$m=2: \quad \sqrt{(2\cdot 2+1)^2+4} = \sqrt{29} = \left[5;\ \overline{2,\ 1,\ 1,\ 2,\ 10}\right]$$

$$m=3: \quad \sqrt{(2\cdot 3+1)^2+4} = \sqrt{53} = \left[7;\ \overline{3,\ 1,\ 1,\ 3,\ 14}\right]$$

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Como la longitud del período es impar, las ecuaciones $\boldsymbol{x^2-(n^2+4)y^2=-1}$ con n impar, tienen soluciones.

Desarrollo en fracción continua de $\sqrt{n^2-4}$, con $n = 2\cdot m+1$

Haciendo $Q = (2m+1)^2 - 4 = 4m^2 + 4m - 3$, tenemos

$$\sqrt{Q} - 2m = \cfrac{1}{\cfrac{1}{\sqrt{Q}-2m}} = \cfrac{1}{\cfrac{\sqrt{Q}+2m}{4m-3}} = \cfrac{1}{1+\cfrac{\sqrt{Q}-(2m-3)}{4m-3}} = \cfrac{1}{1+\cfrac{1}{\cfrac{4m-3}{\sqrt{Q}-(2m-3)}}} =$$

$$\cfrac{1}{1+\cfrac{1}{\cfrac{(4m-3)\left(\sqrt{Q}+(2m-3)\right)}{16m-12}}} = \cfrac{1}{1+\cfrac{1}{\cfrac{\sqrt{Q}+(2m-3)}{4}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{\sqrt{Q}-(2m-1)}{4}}} =$$

$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{\cfrac{4}{\sqrt{Q}-(2m-1)}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{\cfrac{4\left(\sqrt{Q}+(2m-1)\right)}{8m-4}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{\cfrac{\sqrt{Q}+(2m-1)}{2m-1}}}} =$$

$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{\sqrt{Q}-(2m-1)}{2m-1}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{\cfrac{2m-1}{\sqrt{Q}-(2m-1)}}}}} =$$

$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{\cfrac{(2m-1)\left(\sqrt{Q}+(2m-1)\right)}{8m-4}}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{\cfrac{\sqrt{Q}+(2m-1)}{4}}}}} =$$

$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{\sqrt{Q}-(2m-3)}{4}}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{\cfrac{4}{\sqrt{Q}-(2m-3)}}}}}} =$$

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$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{\cfrac{4\left(\sqrt{Q}+(2m-3)\right)}{16m-12}}}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{\cfrac{\sqrt{Q}+(2m-3)}{4m-3}}}}}} =$$

$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{1+\cfrac{\sqrt{Q}-2m}{4m-3}}}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{1+\cfrac{1}{\cfrac{4m-3}{\sqrt{Q}-2m}}}}}}} =$$

$$\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{1+\cfrac{1}{\cfrac{(4m-3)\left(\sqrt{Q}+2m\right)}{4m-3}}}}}}} = \cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{1+\cfrac{1}{4m+\left(\sqrt{Q}-2m\right)}}}}}} \Rightarrow$$

$$\sqrt{Q} = 2m+\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\cfrac{1}{1+\cfrac{1}{4m+\cfrac{1}{1+\cfrac{1}{(m-1)+\cfrac{1}{2+\cfrac{1}{(m-1)+\ldots}}}}}}}}}}$$

$$\sqrt{(2m+1)^2-4} = \left[2m;\ \overline{1,\ m-1,\ 2,\ m-1,\ 1,\ 4m}\right]$$

$$m=1: \quad \sqrt{(2\cdot 1+1)^2-4} = \sqrt{5} = \left[2;\ \overline{1,\ 0,\ 2,\ 0,\ 1,\ 4}\right] = \left[2;\ \overline{4,\ 4}\right] = \left[2;\ \overline{4}\right]$$

$$m=2: \quad \sqrt{(2\cdot 2+1)^2-4} = \sqrt{21} = \left[4;\ \overline{1,\ 1,\ 2,\ 1,\ 1,\ 8}\right]$$

$$m=3: \quad \sqrt{(2\cdot 3+1)^2-4} = \sqrt{45} = \left[6;\ \overline{1,\ 2,\ 2,\ 2,\ 1,\ 12}\right]$$

$$m=4: \quad \sqrt{(2\cdot 4+1)^2-4} = \sqrt{77} = \left[8;\ \overline{1,\ 3,\ 2,\ 3,\ 1,\ 16}\right]$$

$$m=5: \quad \sqrt{(2\cdot 5+1)^2-4} = \sqrt{117} = \left[10;\ \overline{1,\ 4,\ 2,\ 4,\ 1,\ 20}\right]$$

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Desarrollo en fracción continua de $\sqrt{(3k+1)^2+(2k+1)}$

Haciendo $Q=(3k+1)^2+(2k+1)=9k^2+8k+2$

$$\sqrt{Q}-(3k+1)=\cfrac{1}{\cfrac{1}{\sqrt{Q}-(3k+1)}}=\cfrac{1}{\cfrac{\sqrt{Q}+(3k+1)}{2k+1}}=\cfrac{1}{2+\cfrac{\sqrt{Q}-(k+1)}{2k+1}}=\cfrac{1}{2+\cfrac{1}{\cfrac{2k+1}{\sqrt{Q}-(k+1)}}}$$

$$=\cfrac{1}{2+\cfrac{1}{\cfrac{(2k+1)\left(\sqrt{Q}+(k+1)\right)}{(2k+1)(4k+1)}}}=\cfrac{1}{2+\cfrac{1}{\cfrac{\sqrt{Q}+(k+1)}{4k+1}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{\sqrt{Q}-3k}{4k+1}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\cfrac{4k+1}{\sqrt{Q}-3k}}}}$$

$$=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\cfrac{(4k+1)(\sqrt{Q}+3k)}{2(4k+1)}}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\cfrac{\sqrt{Q}+3k}{2}}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{\sqrt{Q}-3k}{2}}}}$$

$$=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{\cfrac{2}{\sqrt{Q}-3k}}}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{\cfrac{2(\sqrt{Q}+3k)}{2(4k+1)}}}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{1+\cfrac{\sqrt{Q}-(k+1)}{4k+1}}}}}$$

$$=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{1+\cfrac{1}{\cfrac{4k+1}{\sqrt{Q}-(k+1)}}}}}}$$

$$\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{1+\cfrac{1}{\cfrac{(4k+1)\left(\sqrt{Q}+(k+1)\right)}{(2k+1)(4k+1)}}}}}}$$

$$=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{1+\cfrac{1}{2+\cfrac{\sqrt{Q}-(3k+1)}{2k+1}}}}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\cfrac{2k+1}{\sqrt{Q}-(3k+1)}}}}}}}$$

$$=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3k+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\cfrac{(2k+1)\left(\sqrt{Q}+(3k+1)\right)}{2k+1}}}}}}}$$

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$$= \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3k + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2(3k+1) + \left(\sqrt{Q} - (3k+1)\right)}}}}}} \Rightarrow$$

$$\boldsymbol{\sqrt{(3k+1)^2 + (2k+1)} = \left[3k+1; \overline{2, 1, 3k, 1, 2, 2(3k+1)}\right]}$$

$k = 1$: $\sqrt{(3 \cdot 1 + 1)^2 + (2 \cdot 1 + 1)} = \sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$

$k = 2$: $\sqrt{(3 \cdot 2 + 1)^2 + (2 \cdot 2 + 1)} = \sqrt{54} = [7; \overline{2, 1, 6, 1, 2, 14}]$

$k = 3$: $\sqrt{(3 \cdot 3 + 1)^2 + (2 \cdot 3 + 1)} = \sqrt{107} = [10; \overline{2, 1, 9, 1, 2, 20}]$

$k = 4$: $\sqrt{(3 \cdot 4 + 1)^2 + (2 \cdot 4 + 1)} = \sqrt{178} = [13; \overline{2, 1, 12, 1, 2, 26}]$

$k = 5$: $\sqrt{(3 \cdot 5 + 1)^2 + (2 \cdot 5 + 1)} = \sqrt{267} = [16; \overline{2, 1, 15, 1, 2, 32}]$

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\sqrt{(3k+2)^2-(2k+1)}$

Haciendo $Q=(3k+2)^2-(2k+1)=9k^2+10k+3$

$$\sqrt{Q}-(3k+1)=\frac{1}{\frac{1}{\sqrt{Q}-(3k+1)}}=\frac{1}{\frac{\sqrt{Q}+(3k+1)}{2(2k+1)}}=\frac{1}{1+\frac{\sqrt{Q}-(k+1)}{2(2k+1)}}$$

$$=\frac{1}{1+\frac{1}{\frac{2(2k+1)}{\sqrt{Q}-(k+1)}}}=\frac{1}{1+\frac{1}{\frac{2(2k+1)\left(\sqrt{Q}+(k+1)\right)}{2(2k+1)^2}}}=\frac{1}{1+\frac{1}{\frac{\sqrt{Q}+(k+1)}{2k+1}}}$$

$$=\frac{1}{1+\frac{1}{2+\frac{\sqrt{Q}-(3k+1)}{2k+1}}}=\frac{1}{1+\frac{1}{2+\frac{1}{\frac{2k+1}{\sqrt{Q}-(3k+1)}}}}$$

$$=\frac{1}{1+\frac{1}{2+\frac{1}{\frac{(2k+1)\left(\sqrt{Q}+(3k+1)\right)}{2(2k+1)}}}}=\frac{1}{1+\frac{1}{2+\frac{1}{\frac{\sqrt{Q}+(3k+1)}{2}}}}$$

$$=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{\sqrt{Q}-(3k+1)}{2}}}}=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{\frac{2}{\sqrt{Q}-(3k+1)}}}}}$$

$$=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{\frac{2\left(\sqrt{Q}+(3k+1)\right)}{2(2k+1)}}}}}=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{2+\frac{\sqrt{Q}-(k+1)}{2k+1}}}}}$$

$$=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{2+\frac{1}{\frac{2k+1}{\sqrt{Q}-(k+1)}}}}}}=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{2+\frac{1}{\frac{(2k+1)\left(\sqrt{Q}+(k+1)\right)}{2(2k+1)^2}}}}}}$$

$$=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{2+\frac{1}{1+\frac{\sqrt{Q}-(3k+1)}{2(2k+1)}}}}}}=\frac{1}{1+\frac{1}{2+\frac{1}{(3k+1)+\frac{1}{2+\frac{1}{1+\frac{1}{\frac{2(2k+1)}{\sqrt{Q}-(3k+1)}}}}}}}$$

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$$= \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{(3k+1)+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\cfrac{2(2k+1)\left(\sqrt{Q}+(3k+1)\right)}{2(2k+1)}}}}}}}$$

$$= \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{(3k+1)+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2(3k+1)+\left(\sqrt{Q}-(3k+1)\right)}}}}}} \Rightarrow$$

$$\boldsymbol{\sqrt{(3k+2)^2-(2k+1)} = \left[3k+1;\overline{1,2,3k+1,2,1,2(3k+1)}\right]}$$

$k=1$: $\sqrt{(3\cdot 1+2)^2-(2\cdot 1+1)} = \sqrt{22} = [4;\overline{1,2,4,2,1,8}]$

$k=2$: $\sqrt{(3\cdot 2+2)^2-(2\cdot 2+1)} = \sqrt{59} = [7;\overline{1,2,7,2,1,14}]$

$k=3$: $\sqrt{(3\cdot 3+2)^2-(2\cdot 3+1)} = \sqrt{114} = [10;\overline{1,2,10,2,1,20}]$

$k=4$: $\sqrt{(3\cdot 4+2)^2-(2\cdot 4+1)} = \sqrt{187} = [13;\overline{1,2,13,2,1,26}]$

$k=5$: $\sqrt{(3\cdot 5+2)^2-(2\cdot 5+1)} = \sqrt{278} = [16;\overline{1,2,16,2,1,32}]$

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\boldsymbol{\sqrt{(6k-1)^2+(4k-1)}}$

Haciendo $Q=(\boldsymbol{6k-1})^{\boldsymbol{2}}+(\boldsymbol{4k-1})=4k(9k-2)$

$$\sqrt{Q}-(6k-1)=\cfrac{1}{\cfrac{1}{\sqrt{Q}-(6k-1)}}=\cfrac{1}{\cfrac{\sqrt{Q}+(6k-1)}{4k-1}}=\cfrac{1}{3+\cfrac{\sqrt{Q}-(6k-2)}{4k-1}}$$

$$=\cfrac{1}{3+\cfrac{1}{\cfrac{4k-1}{\sqrt{Q}-(6k-2)}}}=\cfrac{1}{3+\cfrac{1}{\cfrac{(4k-1)\left(\sqrt{Q}+(6k-2)\right)}{4(4k-1)}}}=\cfrac{1}{3+\cfrac{1}{(3k-1)+\cfrac{\sqrt{Q}-(6k-2)}{4}}}$$

$$=\cfrac{1}{3+\cfrac{1}{(3k-1)+\cfrac{1}{\cfrac{4}{\sqrt{Q}-(6k-2)}}}}=\cfrac{1}{3+\cfrac{1}{(3k-1)+\cfrac{1}{\cfrac{4\left(\sqrt{Q}+(6k-2)\right)}{4(4k-1)}}}}$$

$$=\cfrac{1}{3+\cfrac{1}{(3k-1)+\cfrac{1}{3+\cfrac{\sqrt{Q}-(6k-1)}{4k-1}}}}=\cfrac{1}{3+\cfrac{1}{(3k-1)+\cfrac{1}{3+\cfrac{1}{\cfrac{4k-1}{\sqrt{Q}-(6k-1)}}}}}$$

$$=\cfrac{1}{3+\cfrac{1}{(3k-1)+\cfrac{1}{3+\cfrac{1}{\sqrt{Q}+(6k-1)}}}}=3+\cfrac{1}{(3k-1)+\cfrac{1}{3+\cfrac{1}{2(6k-1)+\left(\sqrt{Q}-(6k-1)\right)}}}\Rightarrow$$

$$\boldsymbol{\sqrt{(6k-1)^2+(4k-1)}=[6k-1;\ \overline{3,3k-1,3,2(6k-1)}]}$$

$k=1$: $\quad\sqrt{(6\cdot 1-1)^2+(4\cdot 1-1)}=\sqrt{28}=[5;\overline{3,2,3,10}]$

$k=2$: $\quad\sqrt{(6\cdot 2-1)^2+(4\cdot 2-1)}=\sqrt{128}=[11;\overline{3,5,3,22}]$

$k=3$: $\quad\sqrt{(6\cdot 3-1)^2+(4\cdot 3-1)}=\sqrt{300}=[17;\overline{3,8,3,34}]$

$k=4$: $\quad\sqrt{(6\cdot 4-1)^2+(4\cdot 4-1)}=\sqrt{544}=[23;\overline{3,11,3,46}]$

$k=5$: $\quad\sqrt{(6\cdot 5-1)^2+(4\cdot 5-1)}=\sqrt{860}=[29;\overline{3,14,3,58}]$

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Desarrollo en fracción continua de $\sqrt{(\boldsymbol{5k}+\mathbf{1})^{\mathbf{2}}+(\boldsymbol{4k}+\mathbf{1})}$

Haciendo $Q=(\boldsymbol{5k}+\mathbf{1})^{\mathbf{2}}+\boldsymbol{4k}+\mathbf{1}=25k^2+14k+2$

$$\sqrt{Q}-(5k+1)=\cfrac{1}{\cfrac{1}{\sqrt{Q}-(5k+1)}}=\cfrac{1}{\cfrac{\sqrt{Q}+(5k+1)}{4k+1}}=\cfrac{1}{2+\cfrac{\sqrt{Q}-(3k+1)}{4k+1}}$$

$$=\cfrac{1}{2+\cfrac{1}{\cfrac{4k+1}{\sqrt{Q}-(3k+1)}}}=\cfrac{1}{2+\cfrac{1}{\cfrac{(4k+1)\left(\sqrt{Q}+(3k+1)\right)}{(4k+1)^2}}}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{\sqrt{Q}-(5k+1)}{4k+1}}}$$

$$=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cfrac{4k+1}{\sqrt{Q}-(5k+1)}}}}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cfrac{(4k+1)\left(\sqrt{Q}+(5k+1)\right)}{4k+1}}}}$$

$$=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2(5k+1)+\left(\sqrt{Q}-(5k+1)\right)}}}\Rightarrow$$

$$\sqrt{(\boldsymbol{5k}+\mathbf{1})^{\mathbf{2}}+(\boldsymbol{4k}+\mathbf{1})}=[\boldsymbol{5k}+\mathbf{1};\ \overline{\mathbf{2},\mathbf{2},\mathbf{2}(\boldsymbol{5k}+\mathbf{1})}]$$

$k=1$: $\sqrt{(5\cdot 1+1)^2+(4\cdot 1+1)}=\sqrt{41}=[6;\overline{2,2,12}]$

$k=2$: $\sqrt{(5\cdot 2+1)^2+(4\cdot 2+1)}=\sqrt{130}=[11;\overline{2,2,22}]$

$k=3$: $\sqrt{(5\cdot 3+1)^2+(4\cdot 3+1)}=\sqrt{269}=[16;\overline{2,2,32}]$

$k=4$: $\sqrt{(5\cdot 4+1)^2+(4\cdot 4+1)}=\sqrt{458}=[21;\overline{2,2,42}]$

$k=5$: $\sqrt{(5\cdot 5+1)^2+(4\cdot 5+1)}=\sqrt{697}=[26;\overline{2,2,52}]$

Como la longitud del período es impar, las ecuaciones $\boldsymbol{x^2-Qy^2=-1}$ tienen soluciones.

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\sqrt{(17k+2)^2+(8k+1)}$

Haciendo $Q=(\mathbf{17k+2})^{\mathbf{2}}+\mathbf{8k+1}=289k^2+76k+5$

$$\sqrt{Q}-(17k+2)=\frac{1}{\dfrac{1}{\sqrt{Q}-(17k+2)}}=\frac{1}{\dfrac{\sqrt{Q}+(17k+2)}{8k+1}}=\frac{1}{4+\dfrac{\sqrt{Q}-(15k+2)}{8k+1}}$$

$$=\frac{1}{4+\dfrac{1}{\dfrac{8k+1}{\sqrt{Q}-(15k+2)}}}=\frac{1}{4+\dfrac{1}{\dfrac{(8k+1)\left(\sqrt{Q}+(15k+2)\right)}{(8k+1)^2}}}=\frac{1}{4+\dfrac{1}{4+\dfrac{\sqrt{Q}-(17k+2)}{8k+1}}}$$

$$=\frac{1}{4+\dfrac{1}{4+\dfrac{1}{\dfrac{8k+1}{\sqrt{Q}-(17k+2)}}}}=\frac{1}{4+\dfrac{1}{4+\dfrac{1}{\dfrac{(8k+1)\left(\sqrt{Q}+(17k+2)\right)}{8k+1}}}}$$

$$=\frac{1}{4+\dfrac{1}{4+\dfrac{1}{2(17k+2)+\left(\sqrt{Q}-(17k+2)\right)}}}\Rightarrow$$

$$\sqrt{(\mathbf{17k+2})^{\mathbf{2}}+(\mathbf{8k+1})}=[\mathbf{17k+2};\ \overline{\mathbf{4,4,2(17k+2)}}]$$

$k=1$: $\sqrt{(17\cdot 1+2)^2+(8\cdot 1+1)}=\sqrt{370}=[19;\overline{4,4,38}]$
$k=2$: $\sqrt{(17\cdot 2+2)^2+(8\cdot 2+1)}=\sqrt{1313}=[36;\overline{4,4,72}]$
$k=3$: $\sqrt{(17\cdot 3+2)^2+(8\cdot 3+1)}=\sqrt{2834}=[53;\overline{4,4,106}]$
$k=4$: $\sqrt{(17\cdot 4+2)^2+(8\cdot 4+1)}=\sqrt{4933}=[70;\overline{4,4,140}]$
$k=5$: $\sqrt{(17\cdot 5+2)^2+(8\cdot 5+1)}=\sqrt{7610}=[87;\overline{4,4,174}]$

Como la longitud del período es impar, las ecuaciones $\boldsymbol{x^2-Qy^2=-1}$ tienen soluciones.

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\sqrt{(37k+3)^2+(12k+1)}$

Haciendo $Q=(\mathbf{37k+3})^{\mathbf{2}}+\mathbf{12k+1}=1369k^2+234k+10$

$$\sqrt{Q}-(37k+3)=\frac{1}{\frac{1}{\sqrt{Q}-(37k+3)}}=\frac{1}{\frac{\sqrt{Q}+(37k+3)}{12k+1}}=\frac{1}{6+\frac{\sqrt{Q}-(35k+3)}{12k+1}}$$

$$=\frac{1}{6+\frac{1}{\frac{12k+1}{\sqrt{Q}-(35k+3)}}}=\frac{1}{6+\frac{1}{\frac{(12k+1)\left(\sqrt{Q}+(35k+3)\right)}{(12k+1)^2}}}=\frac{1}{6+\frac{1}{6+\frac{\sqrt{Q}-(37k+3)}{12k+1}}}$$

$$=\frac{1}{6+\frac{1}{6+\frac{1}{\frac{12k+1}{\sqrt{Q}-(37k+3)}}}}=\frac{1}{6+\frac{1}{6+\frac{1}{\frac{(12k+1)\left(\sqrt{Q}+(37k+3)\right)}{12k+1}}}}$$

$$=\frac{1}{6+\frac{1}{6+\frac{1}{2(37k+3)+\left(\sqrt{Q}-(37k+3)\right)}}}\Rightarrow$$

$$\sqrt{(\mathbf{37k+3})^{\mathbf{2}}+(\mathbf{12k+1})}=[\mathbf{37k+3};\ \overline{\mathbf{6},\mathbf{6},\mathbf{2}(\mathbf{37k+3})}]$$

$k=1$: $\sqrt{(37\cdot 1+3)^2+(12\cdot 1+1)}=\sqrt{1613}=[40;\overline{6,6,80}]$
$k=2$: $\sqrt{(37\cdot 2+3)^2+(12\cdot 2+1)}=\sqrt{5954}=[77;\overline{6,6,154}]$
$k=3$: $\sqrt{(37\cdot 3+3)^2+(12\cdot 3+1)}=\sqrt{13033}=[114;\overline{6,6,228}]$
$k=4$: $\sqrt{(37\cdot 4+3)^2+(12\cdot 4+1)}=\sqrt{22850}=[151;\overline{6,6,302}]$
$k=5$: $\sqrt{(37\cdot 5+3)^2+(12\cdot 5+1)}=\sqrt{35405}=[188;\overline{6,6,376}]$

Como la longitud del período es impar, las ecuaciones $\boldsymbol{x^2-Qy^2=-1}$ tienen soluciones.

Ignacio Larrosa Cañestro

Desarrollo en fracción continua de $\sqrt{((4m^2+1)k+m)^2+(4mk+1)}$
Es una generalización de los tres últimos apartados.

Haciendo $Q=((4m^2+1)k+m)^2+(4mk+1)$

$$\sqrt{Q}-\left((4m^2+1)k+m\right)=\cfrac{1}{\cfrac{1}{\sqrt{Q}-\left((4m^2+1)k+m\right)}}$$

$$=\cfrac{1}{\cfrac{\sqrt{Q}+((4m^2+1)k+m)}{4mk+1}}=\cfrac{1}{2m+\cfrac{\sqrt{Q}-\left((4m^2-1)k+m\right)}{4mk+1}}$$

$$=\cfrac{1}{2m+\cfrac{1}{\cfrac{4mk+1}{\sqrt{Q}-\left((4m^2-1)k+m\right)}}}=\cfrac{1}{2m+\cfrac{1}{\cfrac{(4mk+1)\left(\sqrt{Q}+\left((4m^2-1)k+m\right)\right)}{(4mk+1)^2}}}$$

$$=\cfrac{1}{2m+\cfrac{1}{2m+\cfrac{\sqrt{Q}-((4m^2+1)k+m)}{4mk+1}}}$$

$$=\cfrac{1}{2m+\cfrac{1}{2m+\cfrac{1}{\cfrac{4mk+1}{\sqrt{Q}-((4m^2+1)k+m)}}}}$$

$$=\cfrac{1}{2m+\cfrac{1}{2m+\cfrac{1}{\cfrac{(4mk+1)\left(\sqrt{Q}+((4m^2+1)k+m)\right)}{4mk+1}}}}$$

$$=\cfrac{1}{2m+\cfrac{1}{2m+\cfrac{1}{2((4m^2+1)k+m)+\left(\sqrt{Q}-((4m^2+1)k+m)\right)}}}\Rightarrow$$

$$\sqrt{((4m^2+1)k+m)^2+(4mk+1)}=[(4m^2+1)k+m;\ \overline{2m,2m,2((4m^2+1)k+m)}]$$

$k=1, m=1$: $\sqrt{41}=[6;\overline{2,2,12}]$
$k=1, m=2$: $\sqrt{370}=[19;\overline{4,4,38}]$
$k=1, m=3$: $\sqrt{1613}=[40;\overline{6,6,80}]$
$k=1, m=4$: $\sqrt{4778}=[69;\overline{8,8,138}]$
$k=1, m=5$: $\sqrt{11257}=[106;\overline{10,10,212}]$

Como la longitud del período es impar, las ecuaciones $x^2-Qy^2=-1$ tienen soluciones.

Ignacio Larrosa Cañestro